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ROTARY INERTIA, AXIAL AND SHEAR DEFORMATION EFFECTS ON THE IN-PLANE NATURAL FREQUENCIES OF SYMMETRIC CROSS-PLY LAMINATED CIRCULAR ARCHES

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The in-plane free vibrational analysis of symmetric cross-ply laminated circular arches is studied using both the Timoshenko and Bernoulli–Euler beam theories. The free vibration equations are derived based on the distributed parameter model. The transfer matrix method is used in the analysis. The numerical algorithm available in the literature is adopted to compute the exact overall dynamic transfer matrix of the curved beam. The rotary inertia, axial and shear deformation effects are considered in the Timoshenko analysis by the first-order shear deformation theory. All these effects are neglected in the Bernoulli–Euler analysis. Radius of the arch/thickness ratios from 5 to 25, fixed–fixed, fixed–simple and fixed–free boundary conditions, and two values of the opening angles (10° and 90°) are taken into consideration in the parametric study. The effects of the ratio of the extensional modulus to the transverse modulus on the natural frequencies are examined. Non-dimensional numerical results are also presented in terms of relative errors.

1. INTRODUCTION

Curved beams are used in many engineering applications. Chidamparam and Leissa [1] summarized the extensive published literature on the vibrations of isotropic and curved bars, rings and arches of arbitrary shape which lie in the plane. However, the dynamic problems of laminated composite curved beams have not been studied extensively. Earlier works are related to the sandwich beams or closed composite rings [2–6].

Bhimaraddi *et al.* [7] presented a 24-d.o.f. of isoparametric finite element for the analysis of generally laminated curved beams. The rotary inertia and shear deformation effects were considered in this study [7]. They gave the natural frequencies of (0°/90°) laminated cantilever thin and thick curved beams. The incremental equations of motion based on the principle of virtual displacements of a continuous medium are formulated using the total Lagrangian description by Liao and Reddy [8]. They developed a degenerate shell element with a degenerate curved beam element as a stiffener for the geometric non-linear analysis of laminated, anisotropic, stiffened shells. Qatu [9] presented a set of in-plane

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equations and their solutions for laminated shallow arches having simply supported boundary conditions. The effects of axial and shear deformations and the rotary inertia were neglected in reference [9]. Qatu and Elsharkawy [10] worked out the in-plane free vibration of anti-symmetric laminated arches with deep curvature and arbitrary boundaries. In-plane free vibration analysis of moderately thick laminated circular beams was studied by Qatu [11]. Yıldırım [12], recently presented non-dimensional in-plane free vibrational characteristics of circular arches considering all the parameters affecting natural frequencies.

As it is well known that the Bernoulli–Euler beam theory does not include the rotary, shear and extensional deformation effects in the vibration analysis. Since the ratio of extensional stiffness to the transverse shear stiffness is high for laminated beams, the effect of the shear deformation in laminated beams is more significant than in homogeneous ones. The study of Abramovich [13], Khdeir and Reddy [14], and Yıldırım *et al.* [15] showed that these effects are more important for straight beams. Abramovich [13] studied these effects for the axial and out-of-plane bending oscillations of the simply supported straight beams. Khdeir and Reddy [14] considered the fundamental frequencies for axial and out-of-plane bending vibrations. Yıldırım *et al.* [15] presented a detailed analysis for the in-plane free vibration behaviour of straight beams.

The main objective of the present study is to present the rotary inertia, axial and shear-deformation effects on the in-plane natural frequencies of curved beams. In this study, the transfer matrix method is preferred as a numerical solution technique for the free vibration analysis of the continuous arch system. The transfer matrix method is used for the solution of one-dimensional problems. However, the method has not been applied widely to the composite analysis of beams. It is well known that the transfer matrix method gives exact results when the exact overall dynamic transfer matrix can be obtained. The main problem with this method is the determination of the exact overall transfer matrix considering rotary inertia, axial and shear deformation effects. In this study, the exact overall dynamic transfer matrix of the arch is obtained using Yıldırım's numerical algorithm, which was successfully used for both isotropic [16–20] and anisotropic [12, 15] materials. The rotary inertia, axial and shear deformation effects have been studied considering R/h (the radius of the arch/thickness) ratios, the boundary conditions, the opening angles, α, and the ratio of extensional modulus to the transverse modulus (E_1/E_2) for the first six free vibration frequencies. The numerical results are given in graphical and tabular forms.

2. FORMULATION OF THE PROBLEM

Yıldırım [21] presented the free and forced vibration equations of initially twisted laminated composite space rods. Referring to this study, the following equations are obtained for the in-plane free vibration of symmetric cross-ply laminated circular beams:

$$\frac{\mathrm{d}U_t}{\mathrm{d}s} = (1/R)U_n + A'_{11}T_t = (1/R)U_n + ADE,$$

$$\frac{\mathrm{d}U_n}{\mathrm{d}s} = -(1/R)U_t + \Omega_b + k'A'_{22}T_n = -(1/R)U_t + \Omega_b + SDE,$$

$$\frac{\mathrm{d}\Omega_b}{\mathrm{d}s} = D'_{33}M_b,$$

$$\frac{\mathrm{d}T_t}{\mathrm{d}s} = (1/R)T_n - \bar{A}\omega^2 U_t,$$

$$\frac{\mathrm{d}T_n}{\mathrm{d}s} = -(1/R)T_t - \bar{A}\omega^2 U_n,$$

$$\frac{\mathrm{d}M_b}{\mathrm{d}s} = -T_n - \bar{I}_3\omega^2\Omega_b = -T_n + RIE. \tag{1}$$

The following assumptions are used in the derivation of equations (1): the relationship between the forces and deformations are small and linear. The bar is made of an elastic, homogeneous and orthotropic material. Cross-sectional area is uniform. Warping and pre-twisting are neglected. The normal, **n**, and binormal, **b**, axes are the principal axes of the beam (Figure 1).

In equations (1), T_t and T_n are the axial and shear forces, M_b is the bending moment, respectively. U_t and U_n are the displacements in the **t** (tangential unit vector) and **n** directions, respectively. Ω_b is the rotation about **b** axis, k' is the shear coefficient factor, ω (rad/s) is the circular frequency and ds ($= R\,\mathrm{d}\theta$) is the infinitesimal length of the beam. ADE, SDE and RIE represent the effects of the axial and shear deformations and the rotary inertia, respectively. Other terms in equations (1) are as follows for laminated beams:

$$\bar{A} = \sum_{k=1}^{N} \rho^{(k)}A^{(k)}, \quad \bar{I}_3 = \sum_{k=1}^{N} \rho^{(k)}I_b^{(k)}, \tag{2}$$

where N represents the total number of plies, and ρ denotes the density of the material. I_b is the moment of inertia about **b** axis and A is the undeformed cross-sectional area. The cross-sectional rigidities in equations (1) are achieved as in the following:

$$A'_{11} = 1\Big/\sum_{k=1}^{N} \bar{Q}_{11}^{(k)}A^{(k)}, \qquad A'_{22} = 1\Big/\sum_{k=1}^{N} \bar{Q}_{22}^{(k)}A^{(k)}, \qquad D'_{33} = 1\Big/\sum_{k=1}^{N} (\bar{Q}_{11}^{(k)}I_b^{(k)}), \tag{3}$$

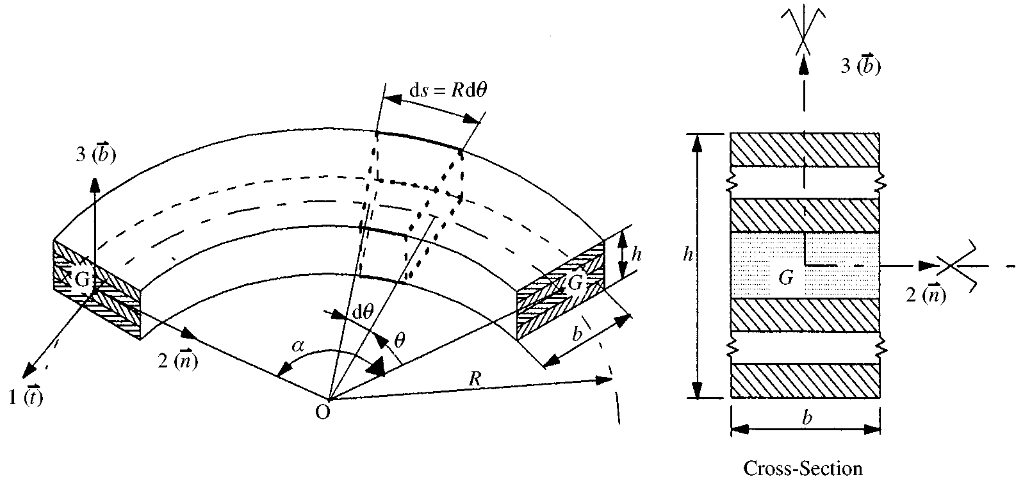

Fig. 1. A laminated composite circular beam and Frenet co-ordinates (**t**, **n**, **b**).

In the above equations, the elements of the transformed reduced stiffness matrix $\bar{\mathbf{Q}}$ for a lamina is obtained as follows:

$$\bar{Q}_{11} = (C'_{11}S'_{11} + C'_{12}S'_{12} + C'_{13}S'_{13})/S'_{11}, \qquad \bar{Q}_{22} = C'_{66}, \tag{4}$$

where C'_{ij} and S'_{ij} denote the elements of the transformed stiffness and compliance matrices, respectively. They are given by Yıldırım [21] for transversely isotropic material as

$$\begin{aligned}
C'_{11} &= m^4 C_{11} + 2(m^2 - m^4)C_{12} + C_{22}(1 - 2m^2 + m^4) + 4(m^2 - m^4)C_{66},\\
C'_{12} &= (m^2 - m^4)C_{11} + (m^2 - m^4)C_{22} + C_{12}(1 - 2m^2 + 2m^4) - 4(m^2 - m^4)C_{66},\\
C'_{13} &= m^2 C_{12} + (1 - m^2)C_{23},\\
C'_{66} &= (m^2 - m^4)C_{11} - 2(m^2 - m^4)C_{12} + (m^2 - m^4)C_{22} + (1 - 4m^2 + 4m^4)C_{66},\\
S'_{11} &= m^4 S_{11} + 2(m^2 - m^4)S_{12} + S_{22}(1 - 2m^2 + m^4) + (m^2 - m^4)S_{66},\\
S'_{12} &= (m^2 - m^4)S_{11} + (m^2 - m^4)S_{22} + S_{12}(1 - 2m^2 + 2m^4) + (m^4 - m^2)S_{66},\\
S'_{13} &= m^2 S_{12} + (1 - m^2)S_{23},
\end{aligned} \tag{5}$$

where

$$m = \cos\beta. \tag{6}$$

and β is the angle between the beam axis and the material symmetry axis (Figure 2). In reference [21], three-dimensional generalized Hooke's law is used based on the classical beam theory to obtain the resultant constitutive equations of the beam.

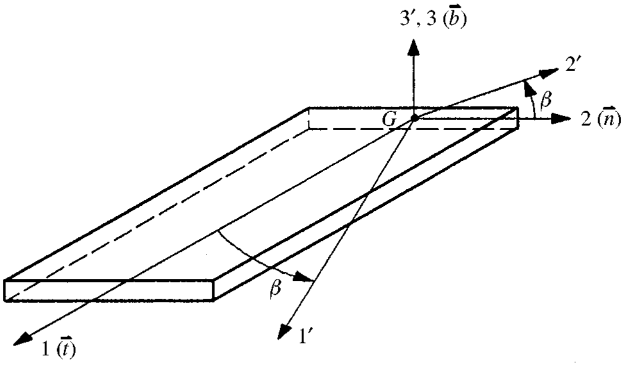

Fig. 2. Positive rotation of principal material axes (1′, 2′, 3′) from the beam (1, 2, 3) axes.

This formulation, which comprises the Poisson effect, coincides with the elasticity and compliance matrices in the off-axis co-ordinate system given by Graff and Springer [22].

The matrix form of equations (1) is as follows:

$$\mathrm{d}\mathbf{Z}/\mathrm{d}s = \mathbf{D}^{*}\mathrm{Z}, \tag{7}$$

where $\mathbf{D}^{*}$ is the dynamic differential matrix and $\mathbf{Z}$ is the state vector. In the transfer matrix method, which provides an exact solution to the problem, the solution of equation (7) is given by Pestel and Leckie [23] as

$$\mathbf{Z}(s) = \mathbf{F}(s, \omega)\mathbf{Z}(0), \tag{8}$$

where $\mathbf{F}$ is the overall dynamic transfer matrix. The standard expression of $\mathbf{F}$ for constant sections is in the following form [23]:

$$\mathbf{F}(s, \omega) = \mathrm{e}^{\mathbf{D}^{*}s} = \mathbf{I} + s\mathbf{D}^{*} + s^{2}\mathbf{D}^{*2}/2! + s^{3}\mathbf{D}^{*3}/3! + \cdots \tag{9}$$

where $\mathbf{I}$ is the unit matrix. In this study, equation (10), which is the other expression of equation (9) based on the Cayley–Hamilton theorem, is used to calculate the transfer matrix by the effective numerical algorithm available in literature [16–20]:

$$\mathbf{F}(s, \omega) = \sum_{k=0}^{5} \Phi_{k}(s, \omega)\mathbf{D}^{*k}, \tag{10}$$

where $\Phi_k(s, \omega)$ are functions of scalar infinite series in s and ω. It is necessary to take a number of terms in the infinite series Φ into consideration to obtain accurate results. The present numerical algorithm [16–20] offers an effective procedure to employ many terms in the calculation of the overall transfer matrix without encountering any ill situations. In this study, 600 terms were taken into account in each Φ series of equation (10). Six hundred terms in equation (10) correspond to 3600 terms in equation (9). After computation of the overall dynamic transfer matrix, the eigenvalue equation can be obtained considering the boundary conditions given at both ends ($s = 0$ and $s = R\alpha$) using equation (8). The boundary conditions are expressed as follows: Clamped end: $U_t = U_n = \Omega_b = 0$; Hinged end: $U_t = U_n = M_b = 0$; and Free end: $T_t = T_n = M_b = 0$. The natural frequency, is, then, computed by setting the determinant of the coefficient matrix in the frequency equation equal to zero. In this study, all numerical computations were performed using double-precision arithmetic. The natural frequencies were obtained by the method of searching the determinant of the coefficient matrix.

3. NUMERICAL EXAMPLES AND DISCUSSION

The material properties of transversely isotropic materials used in this study are given in Table 1. In Table 1, $E_i G_{ij}$, v_{ij} represent the Young's moduli, shear moduli, and the Poisson ratio for an orthotropic lamina, respectively. The shear correction factor is taken to be $k' = 6/5$. In order to examine the accuracy of the present theory with the reported values, miscellaneous problems were solved for different boundary conditions and material types.

As a test example, the axial and out-of-plane free vibration problem of a graphite-epoxy[1] beam is handled. The reported results presented in Table 2 are for Graphite-epoxy[1] material. Table 3 shows the present results of the same example for different material types. A good agreement is observed on comparing Tables 2 and 3.

The first eight purely in-plane (axial + in-plane bending) free vibration frequencies of the test example are presented in Table 4 for different boundary conditions, h/b ratios and material types. As can be seen from Table 4, non-dimensional natural frequencies increase with decreasing h/b ratios.

A number of examples are solved to investigate the effects of the rotary inertia, axial and shear deformations on the natural frequencies of (0°/90°/0°) laminated beams. All layers are assumed to have the same thickness and the beam is assumed to have orthotropic material properties ($E_1/E_2 = 40$, $G_{12} = 0{\cdot}6E_2$, $G_{23} = 0{\cdot}5E_2$, $v_{12} = 0{\cdot}25$[14]) in the material principal axes. The shape of the cross-section is assumed to be a square ($h/b = 1$). The following is used for the determination of

TABLE 1
Mechanical properties of transversely isotropic materials

Material types	E_1 (GPa)	E_2 (GPa)	$G_{12} = G_{13}$ (GPa)	G_{23} (GPa)	ρ (kg/m^3)	v_{12}
Graphite-epoxy[1] (AS4/3501-6)	144·8	9·65	4·14	3·45	1389·23	0·3
Graphite-epoxy[2] (T300/N5208)	181·0	10·3	7·17	3·433	1600·0	0·28
Kevlar 49-epoxy	76·0	5·56	2·30	1·618	1460·0	0·34

TABLE 2

Non-dimensional axial and out-of-plane bending natural frequencies $[=\omega L^2(\rho/E_1h^2)^{1/2}]$ *of symmetric* $[0°/90°/90°/0°]$ *graphite-epoxy*[1] *beams* ($L/h = 10$ *and* $h/b = 1$) *in the literature*

Mode numbers	Simple–simple		Fixed–free		Fixed–simple	Fixed–fixed	
	[24]	[25]	[24]	[25]	[24]	[24]	[25]
1	2·3189	2·3194	0·8891	0·8819	3·0447	3·7751	3·7576
2	7·0171	7·0029	4·1792	4·0259	7·5593	8·0440	7·8718
3	12·132	12·037	9·1916	9·1085	12·565	12·998	12·573
4	17·301	17·015	–(*)	12·193	17·732	18·165	17·373
5	22·533	21·907	14·384	14·080	23·011	23·502	22·200
6	– (*)	23·337	19·175	19·066	28·430	– (*)	23·337
7	27·881	26·736	25·093	23·938	34·027	28·991	27·254
8	33·396	—	30·620	—	39·838	34·675	—

*Longitudinal modes stated by reference [25].

non-dimensional frequencies:

$$\varpi = \sqrt{\frac{\rho}{E_2h^2}}\,\omega R^2. \tag{11}$$

The relative error between Timoshenko's and Bernoulli's frequencies is determined as (ϖ^T = Timoshenko's frequency, ϖ^B = Bernoulli's frequency):

$$\textit{Relative error} = 100(\varpi^T - \varpi^B)/\varpi^T. \tag{12}$$

Variation of the first six in-plane non-dimensional natural frequencies are presented in Figures 3–5 with varying R/h ratios, boundary conditions, and opening angles. The Timoshenko and Bernoulli solutions, and relative error for Bernoulli theory are shown in Figures 3–5. It is observed from the figures that relative errors increase with decreasing R/h ratios, increasing the number of modes, decreasing opening angles and increasing the number of constraints for boundary conditions. For the fundamental frequencies, while the absolute relative error is 0·8% for the fixed–free beam with $R/h = 25$ and $\alpha = 90°$, this value reaches 16% for the fixed–fixed beam with $R/h = 25$ and $\alpha = 90°$. For the sixth natural frequency of the fixed–fixed beam with $R/h = 5$ and $\alpha = 10°$, the absolute relative error rises by 6,950%. The absolute relative error for the fundamental frequency is equal to 830% for fixed–fixed ends with $R/h = 25$ and $\alpha = 10°$. Figures 3–5 display the application limits of the Bernoulli theory for $L/h \leqslant 25$ for $\alpha = 10°$ and $\alpha = 90°$. The relative errors from the inner opening angles, $10° < \alpha < 90°$, can be estimated approximately depending on the errors for 10° and 90°. It is clearly understood from these figures that the free and forced vibration of laminated composite curved beams must be studied with the shear deformation theories.

TABLE 3

Non-dimensional axial and out-of-plane bending natural frequencies $[=\omega L^2(\rho/E_1 h^2)^{1/2}]$ *of symmetric* [0°/90°/90°/0°] *beams* ($L/h = 10$ *and* $h/b = 1$) *for different material types obtained in the present study.* (SS = *simple–simple*, CC = *clamped–clamped*, CF = *clamped–free*, CS = *clamped–simple*)

	Graphite-epoxy[1]				Graphite-epoxy[2]				Kevlar-epoxy			
Modes	SS	CF	CS	CC	SS	CF	CS	CC	SS	CF	CS	CC
1	2·31426	0·88516	3·01055	3·69637	2·34371	0·89105	3·08802	3·82440	2·31016	0·88443	2·99913	3·67744
2	6·99485	4·11318	7·40702	7·75292	7·20974	4·22560	7·68303	8·08266	6·96305	4·09664	7·36633	7·70440
3	12·0310	8·97506	12·2248	12·4153	12·5279	9·30243	12·7602	12·9858	11·9576	8·92686	12·1460	12·3316
4	17·0109	11·4714	17·1062	17·1959	17·8101	11·4189	17·9269	18·0372	16·8933	11·5021	16·9857	17·0725
5	21·9063	13·9443	21·9572	22·0094	22·8378	14·5493	22·8378	22·8378	21·7451	13·8553	21·7943	21·8449
6	22·9427	18·9405	22·9427	22·9427	23·0059	19·8394	23·0688	23·1329	23·0042	18·8085	23·0042	23·0042
7	26·7379	23·8304	26·7669	26·7945	28·1317	25·0269	28·1679	28·2026	26·5338	23·6551	26·5619	26·5885
8	31·5247	28·6788	31·5420	31·5604	33·2074	30·1665	33·2291	33·2517	31·2787	28·4611	31·2954	31·3132

TABLE 4

The first eight non-dimensional in-plane natural frequencies obtained in the present study $[=\omega L^2(\rho/E_1 h^2)^{1/2}]$ *of symmetric* [0°/90°/90°/0°] *beams* ($L/h = 10$) *for different material types and* h/b *ratios* (CC = *clamped–clamped*, CF = *clamped–free*, CS = *clamped–simple*)

	Graphite-epoxy[1]			Graphite-epoxy[2]			Kevlar-epoxy		
Modes	CF	CS	CC	CF	CS	CC	CF	CS	CC
				$h/b = 2$					
1	0·36653	1·52926	2·12319	0·36602	1·54728	2·17239	0·36770	1·53782	2·13898
2	2·15825	4·55124	5·27546	2·18958	4·69840	5·52258	2·17117	4·59182	5·33496
3	5·55123	8·58203	9·28724	5·74145	9·04211	9·91200	5·60255	8·68751	9·42044
4	9·82708	13·1993	13·7974	10·3673	14·1581	14·9582	9·94965	13·3991	14·0288
5	11·4715	18·1355	18·6090	11·4189	19·7448	20·4267	11·5022	18·4521	18·9565
6	14·6215	22·9428	22·9428	15·6937	22·8378	22·8378	14·8441	23·0043	23·0043
7	19·6852	23·2325	23·5943	21·4361	25·6054	26·1610	20·0293	23·6816	24·0710
8	24·8755	28·3992	28·6720	27·4135	31·6156	32·0573	25·3558	28·9914	29·2872
				$h/b = 1$					
1	0·70980	2·63779	3·38738	0·71474	2·76135	3·63052	0·71323	2·66748	3·43841
2	3·63691	6·89872	7·43496	3·82779	7·47908	8·21125	3·68102	7·01440	7·58090
3	8·33536	11·7972	12·1429	9·03423	13·0805	13·6055	8·47446	12·0355	12·4065
4	11·4715	16·8954	17·0922	11·4189	19·0230	19·3546	11·5022	17·2756	17·4903
5	13·4299	22·0217	22·1391	14·8775	22·8378	22·8378	13·6991	22·5527	22·6818
6	18·6600	22·9428	22·9428	20·9817	25·0656	25·2741	19·0761	23·0043	23·0043
7	23·8492	27·1213	27·1906	27·1115	31·1096	31·2392	24·4197	27·8058	27·8825
8	28·9894	32·1834	32·2274	33·2062	37·1210	37·2041	29·7147	33·0212	33·0697

TABLE 4 (continued)

Modes	Graphite-epoxy[1] CF	Graphite-epoxy[1] CS	Graphite-epoxy[1] CC	Graphite-epoxy[2] CF	Graphite-epoxy[2] CS	Graphite-epoxy[2] CC	Kevlar-epoxy CF	Kevlar-epoxy CS	Kevlar-epoxy CC
					$h/b = 0{\cdot}5$				
1	1·26904	3·71748	4·31837	1·31265	4·10562	4·88290	1·28096	3·79045	4·41853
2	5·00902	8·54610	8·70002	5·52592	9·67737	9·94947	5·10554	8·74522	8·91446
3	10·3700	13·5953	13·6797	11·4189	15·6197	15·7635	10·6038	13·9414	14·0332
4	11·4715	18·6185	18·6378	11·6852	21·5725	21·6146	11·5022	19·1148	19·1366
5	15·4237	22·9428	22·9428	17·6707	22·8378	22·8378	15·8094	23·0043	23·0043
6	20·5166	23·5956	23·6142	23·6914	27·4731	27·5044	21·0529	24·2408	24·2610
7	25·1711	28·5203	28·5316	29·1859	33·2638	33·3117	25·8412	29·3085	29·3227
8	28·9895	29·1985	33·4660	33·6091	33·6532	44·9892	29·7513	29·9279	34·4036

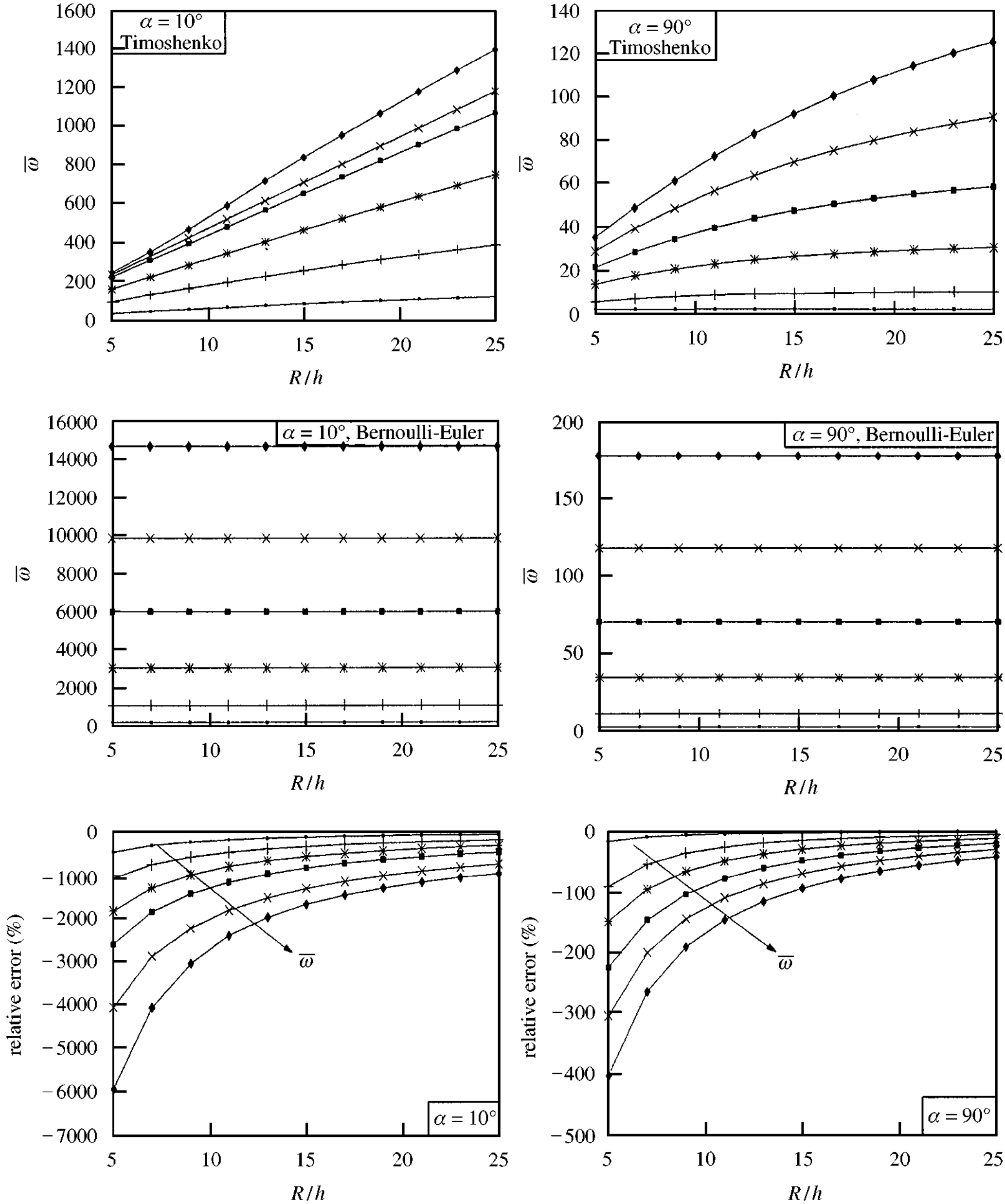

Fig. 3. The first six natural frequencies of the fixed–free circular beam.

The effects of the extensional modulus to the transverse modulus, E_1/E_2, on the fundamental natural frequencies are examined and the results are presented in Table 5. As can be expected, when E_1/E_2 increases, the effects of the rotary inertia, axial and shear deformation effects increase considerably.

Finally, the purely in-plane Timoshenko's and Bernoulli's frequencies of a [0°/90°/90°/0°] Graphite-epoxy straight beam with fixed–free and fixed–simple

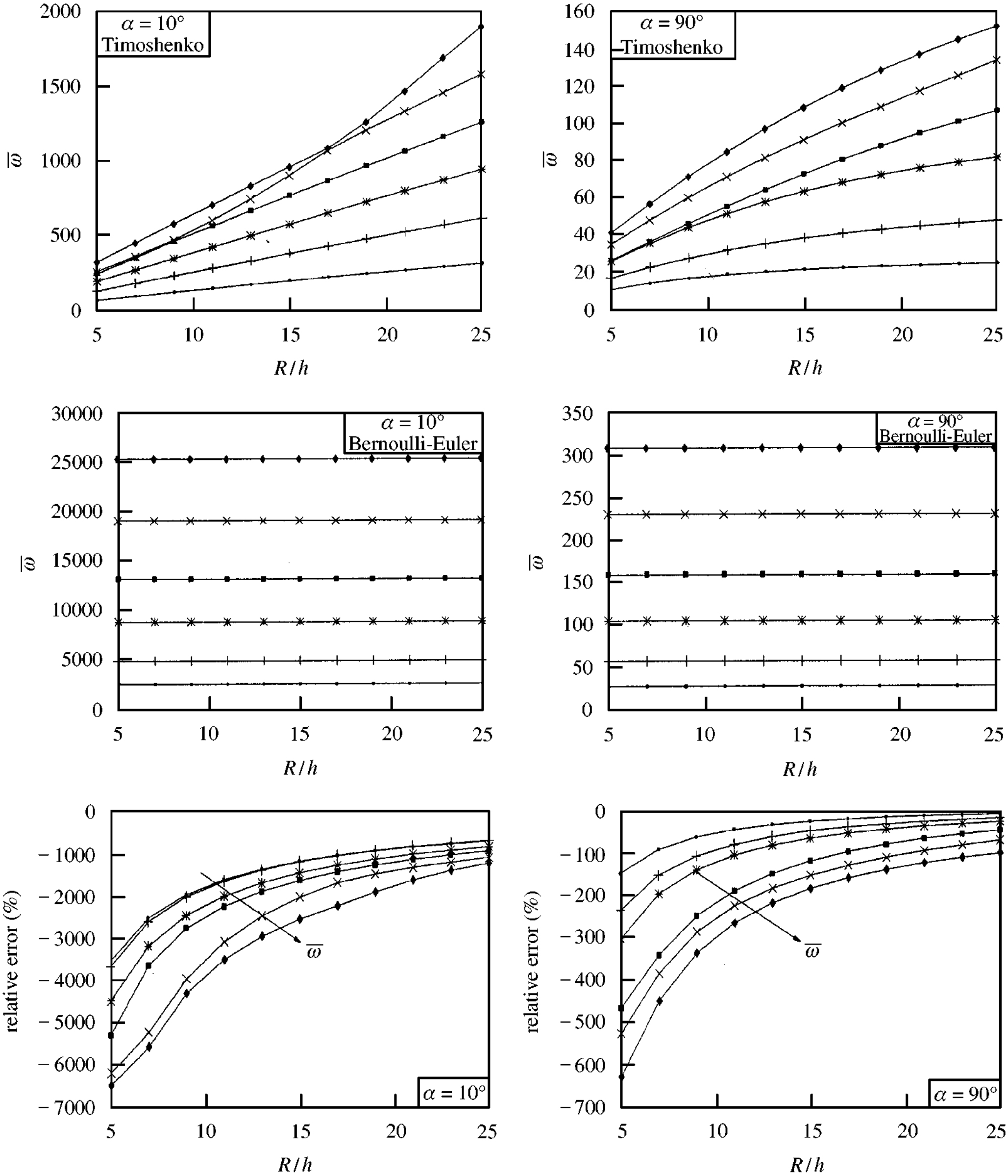

Fig. 4. The first six natural frequencies of the fixed–simple circular beam.

ends ($L/h = 10$, $h/b = 1$) are given in Table 6 in a comparative manner. The rotary inertia, the shear and extensional deformation effects are considerably more important for straight beams than curved beams.

4. CONCLUSIONS

The in-plane free vibration analysis of symmetric cross-ply laminated circular arcs was studied to investigate the axial and shear deformations, and the rotary

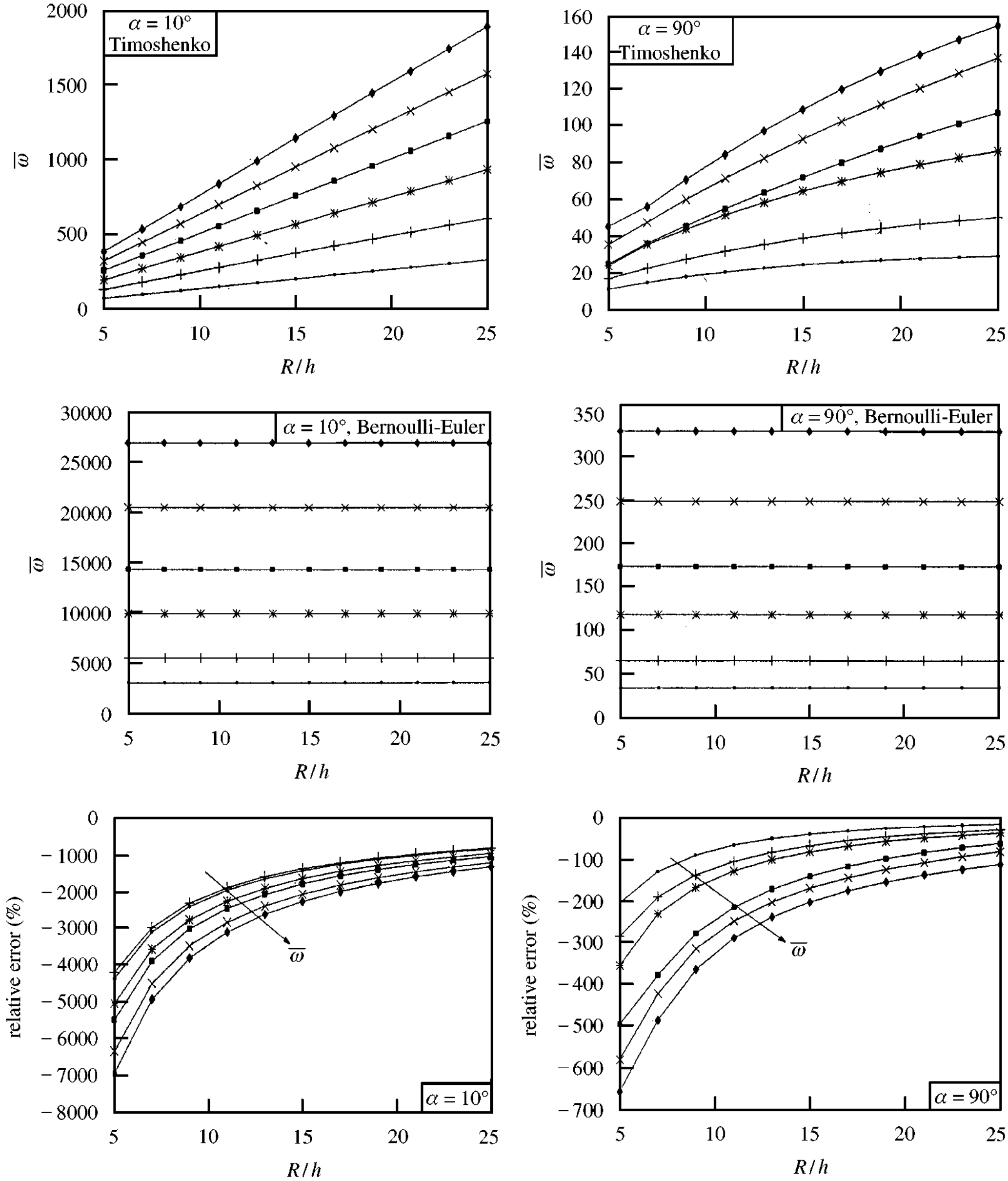

Fig. 5. The first six natural frequencies of the fixed–fixed circular beam.

inertia effects on the first six natural frequencies. The accuracy of the formulation was verified by solving miscellaneous numerical examples. It was observed from the comparisons that this formulation offers reasonable results for the natural frequencies associated with the first and the higher modes. A non-dimensional parametric study was performed based on the Timoshenko and Bernoulli–Euler beam theories for different boundary conditions, slenderness ratios, the ratio of the extensional modulus to the transverse shear modulus, and opening angles. It was

TABLE 5

Variation of the purely fundamental in-plane Timoshenko's natural frequencies $[=\omega L^2(\rho/E_2h^2)^{1/2}]$ *of* $[0°/90°/0°]$ *circular beam with the ratio of* E_1/E_2 *for fixed–fixed ends* ($L/h=10$, $h/b=1$)

α (°)	E_1/E_2	Timoshenko	Relative error for Bernoulli (− %)
10	1	108·31	440
	20	129·89	1561
	40	134·72	2151
90	1	6·217	31
	20	16·759	44
	40	19·273	76

TABLE 6

The purely in-plane frequencies $[=\omega L^2(\rho/E_1h^2)^{1/2}]$ *of* $[0°/90°/90°/0°]$ *Graphite-epoxy*[1] *straight beam* ($L/h=10$, $h/b=1$)

	Fixed–free		Fixed–simple	
Modes	Timoshenko	Bernoulli	Timoshenko	Bernoulli
1	0·710	0·742 (− 4·5%)*	2·638	3·251 (− 23%)*
2	3·637	4·646 (− 28%)*	6·899	10·53 (− 53%)*
3	8·335	13·01 (− 56%)*	11·80	21·98 (− 86%)*
4	11·47	25·49 (− 122%)*	16·90	37·58 (− 122%)*
5	13·43	42·13 (− 214%)*	22·02	57·35 (− 160%)*
6	18·66	62·94 (− 237%)*	22·94	81·28 (− 254%)*

*The relative errors for Bernoulli's results [equation (12)].

concluded that dynamical problems of laminated composite circular arches must be solved considering the rotary inertia, axial and transverse shear deformation effects in the mathematical formulation.

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